Bargaining in Dynamic Games

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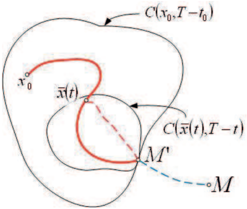
1. Classical control problem.

$$\dot{x} = f(x, u), \quad x \in R^n, u \in U \subset CompR^l,$$
$$x(t_0) = x_0, \quad t \in [t_0, T],$$
$$H(x(T)) = -\rho(x(T), M).$$

$C(x_0, T - t_0)$ – reachability set.
$\bar{x}(t)$ – optimal trajectory.

$\Gamma(x_0, T - t_0), \quad \Gamma(\bar{x}(t), T - t), \quad C(\bar{x}(t), T - t)$



R. Bellmann
Time-consistency, Strong Time-consistency.

2. Multycriterial control.

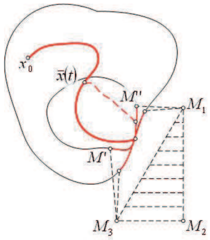
$$\dot{x} = f(x, u), \quad x \in R^n, u \in U \subset CompR^l,$$
$$x(t_0) = x_0, \quad t \in [t_0, T],$$
$$H(x(T)) = \{H_1(x(T)), \ldots, H_k(x(T))\}.$$

Let $k = 3$, $H_i(x(T)) = -\rho(x(T), M_i)$.

Pareto-optimal solution.

$\bar{x}(t)$ – Pareto-optimal trajectory.

$\Gamma(x_0, T - t_0)$, $\Gamma(\bar{x}(t), T - t)$, $C(x_0, T - t_0)$, $C(\bar{x}(t), T - t)$, $P(x_0, T - t_0)$, $P(\bar{x}(t), T - t)$



TC but not STC

3. Nash bargaining solution in Differential Games.

$$\dot{x} = f(x, u_1, \ldots, u_n), \quad x \in R^m, u_i \in U_i \subset CompR^l,$$
$$x(t_0) = x_0, \quad t \in [t_0, T],$$

The payoff of player $i \to H_i(x(T))$,
$\Gamma(x_0, T - t_0)$
$W(x_0, T - t_0; \{i\})$ – the garanteed payoff of player i

NB.

$$\max_{x' \in C(x_0, T - t_0)} \prod_{i=1}^{n} (H_i(x') - W(x_0, T - t_0; \{i\})) = \prod_{i=1}^{n} (H_i(\bar{\bar{x}}) - W(x_0, T - t_0; \{i\}))$$

$$\bar{x}(t), \quad x_0 \to \bar{\bar{x}}, \quad \Gamma(\bar{x}(t), T - t), t \in [t_0, T - t_0], \quad W(\bar{x}(t), T - t; \{i\})$$

$$\max_{x' \in C(\bar{x}(t), T - t)} \prod_{i=1}^{n} (H_i(x') - W(\bar{x}(t), T - t; \{i\})) = \prod_{i=1}^{n} (H_i(\bar{\bar{x}}(\bar{x}(t))) - W(\bar{x}(t), T - t; \{i\}))$$

$\bar{\bar{x}}(\bar{x}(t)) \neq const \neq \bar{x}$
NB, not TC, not STC

Regularization. Here we consider simple case of the game with terminal payoff and prescribed duration.

Denote the payoff function as

$$K_i(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot)) = H_i(x(T)),$$

where $x(t)$ is the solution of problem

$$\dot{x} = f(x, u_1, \dots, u_n) \tag{1}$$

$$x(t_0) = x_0$$

Denote the corresponding differential game by $\Gamma(x_0, T - t_0)$. Consider a Bargaining Problem for each subgame $\Gamma(y, T - t)$, where $y \in C(x_0, t)$, and $C(x_0, t)$ is the reachability set for system (1). We can have in mind as bargaining solution Nash Bargaining (NB), (KS). Suppose we define for each $y \in C(x_0, t)$, $t \in [t_0, T]$ the NB solution and denote it by $D(y, T - t)$.

$$D(y, T - t) = \{H_i(\widetilde{x}(T; y, t))\}$$

Suppose $\widetilde{x}(t)$, $t \in [t_0, T]$ is the bargain trajectory in the game $\Gamma(x_0, T - t_0)$ leading from initial point x_0 to bargaining point $\widetilde{x}(T; x_0, t_0)$. As we mention the NB solution as many other bargaing solutions (including Kalai–Smorodinsky) are time inconsistent, which means that

$$D(\widetilde{x}(t), T - t) \neq const \qquad t \in [t_0, T].$$

This makes difficult the implementation of NB in practice.

Here we purpose the regularization mechanism similar to IDP we used considering the similar time-inconsistency problem in differential cooperative games. To look different from IDP (imputation distribution procedure) we shall call in PDP (payoff distribution procedure).

Introduce the function $g_i(\tau)$, $\tau \in [t_0, T]$ such that

$$D_i(x_0, T - t_0) = H(\widetilde{x}(T; x_0, T - t_0)) = \int_{t_0}^{T} g_i(\tau) d\tau.$$

$g_i(\tau)$ is a new control variable ($\tau \in [t_0, T]$) used to distribute the payoff over the time interval $[t_0, T]$.
Time-consistency condition

$$D_i(x_0, T - t_0) = \int_{t_0}^{t} g_i(\tau) d\tau + D_i(\widetilde{x}(\tau), T - \tau).$$

If function D_i is differentiable we get

$$g_i(t) = -\frac{d}{dt} D_i(\widetilde{x}(\tau), T - t),$$

or

$$g_i(t) = -\sum_{i=1}^{n} \frac{\partial D_i}{\partial x_i} f_i(\widetilde{x}(t), \widetilde{u}_1(t), \dots, \widetilde{u}_n(t)), \qquad i = 1, \dots, n.$$

Kalai–Smorodinsky solution

$$\dot{x} = f(x, u_1, \dots, u_n), \qquad x \in R^m, u_i \in U \subset CompR^l$$

$$x(t_0) = x_0, \qquad t \in [t_0, T].$$

The payoff of player i – $H_i(x(T))$
$\Gamma(x_0, T - t_0)$
$W(x_0, T - t_0; \{i\})$ – the max guaranteed payoff of player i
KS–solution
Consider the set $\aleph(x_0, T - t_0)$ of all possible payoffs of players.

$$\aleph(x_0, T - t_0) = \{H_1(x(T)), \dots, H_n(x(T))\}$$

for all possible $x(T) \in C(x_0, T - t_0)$ where $C(x_0, T - t_0)$ is the reachability set of (1).
Correspondingly denote by $\aleph(y, T - t)$, $y \in C(x_0, T - t_0)$ the set of all possible payoffs of players in subgames starting from y with duration $T - t$ ($\Gamma(y, T - t)$).
Denote by

$$\overline{H}_i = \max_{x(T) \in C(x_0, T - t_0)} H(x(T)).$$

Consider a line L connecting the "status point" W with $\overline{H}$.

The intersection of L and $\aleph(x_0, T - t_0)$ is called Kalai–Smorodinsky solution (KS–solution).

Example. KS–solution is time–inconsistent

$$\dot{x} = u_1 + u_2, \qquad |u_1| \leq 1, |u_2| \leq 1, x \in R^2, u_1, u_2 \in R^2$$
$$x_0 = (6, 3), t \in [0, 2], x = (x_1, x_2)$$
$$H_i(x_0, 2; u_1, u_2) = -x(2), K_2(z_0, 2; u_1, u_2) = -|y(2)|$$

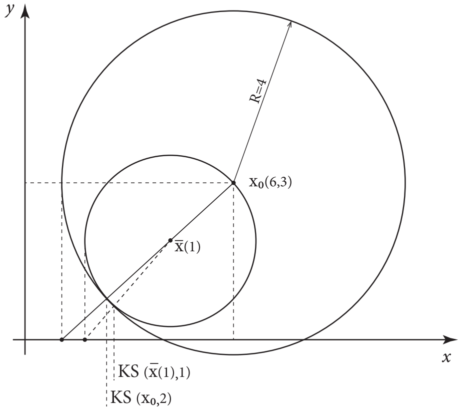
Problem $\Gamma(x_0, 2)$
$\underline{W}(x_0, 2) = (-6, 3)$
$\overline{H}(x_0, 2) = (-2, 0)$
and corresponds to the point $\widehat{x}(x_0, 2) = (2, 0)$, since $\max H_1 = -2$, $\max H_2 = 0$.
$C^2(6, 3; 2)$ is circle with center $(6, 3)$ and radius 4.

Optimal trajectory is to move from $x_0 = (6, 3)$ to $\widehat{x}(x_0, 2) = (2, 0)$ with maximal velocity till the intersection with $C(6, 3; 2)$. The intersection point is KS–solution of $\Gamma(x_0, 2)$.

We see that when moving along KS–solution trajectory $\overline{x}(t)$ connecting x_0 with KS–solution of the problem $\Gamma(x_0, 2)$ in subgame $\Gamma(\overline{x}(1), 1)$ we get a new KS–solution which do not coincide with previous KS–solution of the game $\Gamma(x_0, 2)$.

$$KS(x_0, 2) \neq KS(\overline{x}(1), 1).$$

(time–inconsistency)



4. Differential Cooperative Game.

Differential Cooperative Game $\Gamma(x_0, T - t_0)$ with prescribed duration $T - t_0$ from the initial position x_0.

$$\dot{x} = f(x, u_1, \ldots, u_n), \quad x \in R^n, u_i \in U_i \tag{1}$$

integral payoff

$$K_i(x_0, T - t_0; u_1, \ldots, u_n) = \int_{t_0}^{T} h_i(x(t))dt, \quad h_i > 0, i = 1, \ldots, n.$$

Cooperative form of $\Gamma(x_0, T - t_0)$.

Cooperative behavior $u^*(t) = \{u_1^*(t), \ldots, u_n^*(t)\}$

$$\begin{aligned} \sum_{i=1}^{n} K_i(x_0, T - t_0; u_1^*, \ldots, u_n^*) &= \\ &= \max_{u_1, \ldots, u_n} \sum_{i=1}^{n} K_i(x_0, T - t_0; u_1, \ldots, u_n) = \\ &= \sum_{i=1}^{n} \int_{t_0}^{T} h_i(x^*(t))dt = v(N; x_0, T - t_0), \end{aligned}$$

$x^*(t)$ – cooperative trajectory.

Characteristic Function in $\Gamma(x_0, T - t_0)$.

$v(S; x_0, T - t_0), \quad S \subset N,$

superadditivity: $v(S_1 \cup S_2; x_0, T - t_0) \geq v(S_1; x_0, T - t_0) + v(S_2; x_0, T - t_0)$, $S_1 \cap S_2 = \emptyset$.

There are different ways on how to define c. f.

a. Classical: $v(S; x_0, T - t_0) = Val\Gamma_{S,N\setminus S}(x_0, T - t_0)$, where $\Gamma_{S,N\setminus S}(x_0, T - t_0)$ is a zero-sum game played upon the structure of game $\Gamma(x_0, T - t_0)$ between S as player 1 and $N \setminus S$ as player 2.

b. $v(S; x_0, T - t_0) = \sum_{i \in S} K_i(x_0, T - t_0; \bar{u}_S, \bar{u}_{N\setminus S})$, where $(\bar{u}_S, \bar{u}_{N\setminus S})$ is some giver NE in $\Gamma'_{S,N\setminus S}$ played as non zero-sum game over the structure of $\Gamma(x_0, T - t_0)$ between two players: coalition S as player 1 and $N \setminus S$ as player 2

c. $v(S; x_0, T - t_0) = \max_{u_S = \{u_i, i \in S\}} \sum_{i \in S} K_i(x_0, T - t_0; \bar{u}||u_S)$, where $\bar{u} = (\bar{u}_1, \dots, \bar{u}_n)$ is some fixed NE in $\Gamma(x_0, T - t_0)$.

L. Petrosjan, G. Zaccour Time-consistent Shapley value allocation of pollution cost reduction // Journal of Economics Dunamics & Control, 27 (2003), pp. 381-398.

Let $E(x_0, T - t_0)$ be the imputation set in $\Gamma(x_0, T - t_0)$:

$$E(x_0, T - t_0) = \{\xi = (\xi_i) : \sum_{i=1}^{n} \xi_i = v(N; x_0, T - t_0), \xi_i \geq v(\{i\}; x_0, T - t_0), i \in N\}.$$

Denote by $C^{t-t_0}(x_0), t \in [t_0, T]$ reachable set of the (1).
For each $y \in C^{t-t_0}(x_0)$ consider a subgame $\Gamma(y, T - t)$ of the game $\Gamma(x_0, T - t_0)$, with corresponding c. f. $v(S; y, T - t)$ and set of imputations $E(y, T - t)$.

Definition. A point-to-set mapping $C(y, T - t) \subset E(y, T - t)$ defined for all $y \in C^{t-t_0}(x_0), t \in [t_0, T]$ is call *solution concept* (SC) in the family of subgames $\Gamma(y, T - t)$.

In special cases $C(y, T - t)$ may be a core, NM-solution, Shapley value, nucleous etc.

What happens when the game develops along the cooperative trajectory $x^*(t)$?
We pass through current subgame $\Gamma(x^*(t), T - t)$, wilingly or not updating the current SC $\leftrightarrow C(x^*(t), T - t)$.

Imputation Distribution Procedure (IDP).

Let $\bar{\xi} \in C(x_0, T - t_0)$ and $\beta_i(t), i \in N, t \in [t_0, T]$ satisfies the condition

$$\bar{\xi} = \int_{t_0}^{T} \beta_i(t)dt, \quad i \in N, \quad \beta_i \geq 0.$$

$\beta_i(t)$ is called IDP.

Define

$$\bar{\xi}(\theta) = \int_{t_0}^{\theta} \beta_i(t)dt, \quad i \in N, \quad \beta_i \geq 0.$$

Definition. The SC $C(x^*(t), T - t), t \in [t_0, T]$ is called *time-consistent* (TC) if there exist such IDP $\beta(t) = \{\beta_i(t)\}$ that

$$\bar{\xi} - \bar{\xi}(\theta) \in C(x^*(\theta), T - \theta)$$

for all $\theta \in [t_0, T]$.

Definition. The SC $C(x^*(t), T - t), t \in [t_0, T]$ is called *strongly time-consistent* (STC) if there exist such IDP $\beta(t) = \{\beta_i(t)\}$ that

$$\bar{\xi}(\theta) \oplus \bar{C}(x^*(\theta), T - \theta) \subset C(x_0, T - t_0)$$

for all $\theta \in [t_0, T]$. Here $\bar{\xi} \oplus A$ means the set of all possible vectors $\bar{\xi} + \eta$ for all $\eta \in A$.

Consider $C(x^*(t), T-t)$ along $x^*(t), t \in [t_0, T]$. Suppose we can construct a differentiable selector $\xi^t \in C(x^*(t), T-t)$, then we can easily get for $\beta(t)$ the following formula

$$\bar{\xi} = \bar{\xi}(\theta) + \xi^t \quad \rightarrow \quad \bar{\xi} = \int_{t_0}^{\theta} \beta_i(t) dt + \xi^t$$

$$\beta_i(t) = -\frac{d}{dt} \xi^t$$

If ξ^t can be chosen as monotonic nonincreasing (which is very possible since $h_i > 0$, then $\beta_i \geq 0$, and SC is TC.
If the case (for instance) $C(y, T-t)$ is a Shapley value, we get

$$\beta_i(t) = -\sum_{S \subset N, S \ni i} \frac{(n-s)!(s-1)!}{n!} \left[\frac{d}{dt} v(x^*(t), T-t; S) - \right.$$

$$\left. - \frac{d}{dt} v(x^*(t), T-t; S \setminus \{i\}) \right]$$

and we need only differentiability of the value function (c. f.) $v(x, T-t; S)$.

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